

# 基本超几何级数的求和

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**摘 要** 本文运用基本超几何级数求和的一个简单算法,求得一些基本超几何级数的求和公式.

**关键词** 基本超几何级数;求和; $\Gamma$ 函数

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## 0 引言

设  $a, q$  是复数,且  $0 < q < 1$ , 定义:

$$\begin{aligned}(a)_n &= (a; q)_n = \prod_{i=1}^n (1 - aq^{i-1}), (a)_0 = 1, (a)_\infty \\ &= \prod_{i=1}^\infty (1 - aq^{i-1}), (a_1, a_2, \dots, a_m; q)_\beta = \prod_{i=1}^m (a_i; q)_\beta,\end{aligned}$$

$${}_r\Phi_s \left[ \begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{n=0}^\infty \frac{\prod_{i=1}^r (a_i)_n}{\prod_{i=1}^s (b_i)_n} \cdot ((-1)^n q^{\binom{n}{2}})^{1+s-r} \cdot \frac{z^n}{(q)_n}$$

此处当  $r = 1 + s$ ,  $|z| < 1$ ; 当  $r \leq s$ ,  $|z| < +\infty$

$${}_{r+1}W_r(a; a_1, \dots, a_{r-2}; q, z) = {}_{r+1}\Phi_r \left[ \begin{matrix} a, \sqrt{aq}, -\sqrt{aq}, a_1, \dots, a_{r-2} \\ \sqrt{a}, -\sqrt{a}, aq/a_1, \dots, aq/a_{r-2} \end{matrix}; q, z \right].$$

定义:

$$\begin{aligned}\tau_q(x) &= \begin{cases} \frac{(q)_\infty}{(q^x)_\infty} (1-q)^{1-x} & |q| < 1 \\ \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} (q-1)^{1-x} q^{\binom{x}{2}} & q > 1 \end{cases} \\ \Psi_q(x) &= \frac{d}{dx} \ln |\tau_q(x)|.\end{aligned}$$

当  $0 < q < 1$  时,易推出  $\Psi_q(x) = -\ln(1-q) + \sum_{n=0}^\infty \frac{q^{x+n}/\ln q}{1-q^{x+n}}$ . 文献(1)中,运用了一个简单算法,得到了一些基本超几何级数的求和公式,这个简单算法为

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$$\begin{aligned}
& {}_{r+1}\Phi_{r+1}\left[\begin{matrix} a_1q, a_2q, \dots, a_rq, q \\ b_1q, b_2q, \dots, b_rq, q^2 \end{matrix}; q, q^{1+r-r}z\right] \\
&= \frac{(-1)^{1+r-r}(1-q)}{z} \cdot \frac{\prod_{i=1}^r (1-b_i)}{\prod_{i=1}^r (1-a_i)} \left\{ {}_r\Phi_r\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] - 1 \right\}.
\end{aligned}$$

本文继续使用这个算法,来对一些基本超几何级数进行求和.

## 1 几个求和公式

令  $\widetilde{\Psi}_q(x) = (q-1) \sum_{n=0}^{\infty} \frac{xq^n}{1-xq^n}$  ( $0 < q < 1$ ), 则有

$$\widetilde{\Psi}_q(x) = (q-1) \frac{\ln(1-q)}{\ln q} + \frac{(q-1)}{\ln q} \Psi_q(x).$$

定理 1  $\lim_{N \rightarrow +\infty} (\widetilde{\Psi}_q(\frac{x}{q^N}) - \widetilde{\Psi}_q(\frac{y}{q^N})) = \widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y)$

证明 依据  $\widetilde{\Psi}_q(x)$  的表示, 易推出

$$\widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y) = \sum_{n=0}^{\infty} \frac{q^n(y-x)(1-q)}{(1-xq^n)(1-yq^n)}$$

$$\widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(xq^N) = (q-1) \sum_{n=0}^{N-1} \frac{xq^n}{1-xq^n}$$

$$\begin{aligned}
\text{则 } \widetilde{\Psi}_q\left(\frac{x}{q^N}\right) - \widetilde{\Psi}_q\left(x\right)\left(\frac{y}{q^N}\right) &= (\widetilde{\Psi}_q\left(\frac{x}{q^N}\right) - \widetilde{\Psi}_q(x)) - (\widetilde{\Psi}_q\left(\frac{y}{q^N}\right) - \widetilde{\Psi}_q(y)) + \widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y) \\
&= (q-1) \sum_{n=0}^{N-1} \left[ \frac{\frac{x}{q} \cdot \frac{1}{q^{N-n-1}}}{1 - \frac{x}{q} \cdot \frac{1}{q^{N-n-1}}} - \frac{\frac{y}{q} \cdot \frac{1}{q^{N-n-1}}}{1 - \frac{y}{q} \cdot \frac{1}{q^{N-n-1}}} \right] + \widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y) \\
&= (q-1) \sum_{n=0}^{N-1} \left[ \frac{\frac{x}{q} \cdot \frac{1}{q^n}}{1 - \frac{x}{q} \cdot \frac{1}{q^n}} - \frac{\frac{y}{q} \cdot \frac{1}{q^n}}{1 - \frac{y}{q} \cdot \frac{1}{q^n}} \right] + \widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y) \\
&= \sum_{n=0}^{N-1} \frac{q^n(1-q)(q/x - q/y)}{(1 - q^{n+1}/x)(1 - q^{n+1}/y)} + \widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y) \\
&= \widetilde{\Psi}_q\left(\frac{q}{x}\right) - \widetilde{\Psi}_q\left(\frac{q}{y}\right) + \widetilde{\Psi}_q(x) - \widetilde{\Psi}_q(y).
\end{aligned}$$

定理 2  ${}_3\Phi_2\left[\begin{matrix} aq, q, q \\ bq, q^2 \end{matrix}; q, \frac{b}{a}\right] = \sum_{n=0}^{\infty} \frac{a(1-b)(1-q)q^n}{(1-bq^n)(a-bq^n)}$ .

证明 由文献(1)知

$$\begin{aligned}
{}_4\Phi_3\left[\begin{matrix} aq, cq, q, q \\ bq, acq^2/b, q^2 \end{matrix}; q, q\right] &= \frac{1}{q} \frac{(1-b)(1-qac/b)}{(1-a)(1-c)} \cdot \left\{ \widetilde{\Psi}_q\left(\frac{q}{b}\right) - \widetilde{\Psi}_q\left(\frac{cq}{b}\right) - \widetilde{\Psi}_q\left(\frac{aq}{b}\right) + \right. \\
&\quad \left. \widetilde{\Psi}_q\left(\frac{acq}{b}\right) - \frac{(1-q)(a, c, q, acq^2/b^2; q)_{\infty}}{(b/q, aq/b, cq/b, acq/b; q)_{\infty}} \cdot {}_3\Phi_2\left[\begin{matrix} aq/b, cq/b, q/b \\ q^2/b, acq^2/b^2 \end{matrix}; q, q\right] \right\}
\end{aligned}$$

(其中  $(a_1, \dots, a_m; q)_\infty = \prod_{i=1}^m (a_i, q)_\infty$ )

令  $c = q^{-N}$  且  $N \rightarrow +\infty$ , 由定理 1, 有

$${}_3\Phi_2 \left[ \begin{matrix} aq, q, q \\ bq, q^2 \end{matrix}; q \right] = \frac{(1-1/b)}{(1-1/a)} (\widetilde{\Psi}_q(b) - \widetilde{\Psi}_q(\frac{b}{a}))$$

$$\text{而} \quad \widetilde{\Psi}_q(b) - \widetilde{\Psi}_q(\frac{b}{a}) = \sum_{n=0}^{\infty} \frac{q^n (\frac{b}{a} - b)(1-q)}{(1-bq^n)(1-\frac{b}{a}q^n)}$$

$$= \sum_{n=0}^{\infty} \frac{baq^n (\frac{1}{a} - 1)(1-q)}{(1-bq^n)(a-bq^n)}$$

$$\text{故} \quad {}_3\Phi_2 \left[ \begin{matrix} aq, q, q \\ bq, q^2 \end{matrix}; q, \frac{b}{a} \right] = \sum_{n=0}^{\infty} \frac{a(1-b)(1-q)q^n}{(1-bq^n)(a-bq^n)}.$$

$$\text{定理 3} \quad {}_5\Phi_4 \left[ \begin{matrix} aq, -\sqrt{aq^2}, bq, q, q \\ -\sqrt{aq}, aq^2/b, aq^2, q^2 \end{matrix}; q, \frac{q\sqrt{a}}{b} \right] = \frac{b(aq-b)(1-aq)(1-q)}{(1-b)(1+\sqrt{aq})} \\ \left\{ \sum_{n=0}^{\infty} \left[ \frac{1/b}{(1-\sqrt{aq^{n+1}})(1-aq^{n+1})} - \frac{1}{(b-\sqrt{aq^{n+1}})(b-aq^{n+1})} \right] \right\}.$$

证明 由文献(2), 可得:

$${}_9\Phi_8 \left[ \begin{matrix} \sqrt{aq^2}, -\sqrt{aq^2}, bq, cq, dq, a^2q^2/bcd, aq, q, q \\ \sqrt{aq}, -\sqrt{aq}, aq^2/b, aq^2/c, aq^2/d, bcdq/a, q^2, aq^2 \end{matrix}; q, q \right] \\ = \frac{1}{q} \frac{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq)(1-bcd/a)}{(1-q^2a)(1-b)(1-c)(1-d)(1-qa^2/bcd)} \times \\ \left[ \widetilde{\Psi}_q(\frac{aq}{d}) - \widetilde{\Psi}_q(aq) + \widetilde{\Psi}_q(\frac{aq}{c}) - \widetilde{\Psi}_q(\frac{aq}{cd}) + \widetilde{\Psi}_q(\frac{b}{a}) - \widetilde{\Psi}_q(\frac{bd}{a}) + \widetilde{\Psi}_q(\frac{bcd}{a}) - \widetilde{\Psi}_q(\frac{bc}{a}) - \right. \\ \left. \frac{(1-q)(c, d, q, a^2q/bcd, bq/c, bq/d, bq, b^3cd/a^2; q)_\infty}{(a/b, aq/c, aq/d, bcd/a, bc/a, bd/a, aq/cd, b^2q/a; q)_\infty} \times \right. \\ \left. {}_7\Phi_7(\frac{b^2}{a}; b, \frac{bc}{a}, \frac{bd}{a}, \frac{b}{a}, \frac{aq}{cd}; q, q) \right]$$

令  $d = q^{-N}$  且  $N \rightarrow +\infty$ , 再由定理 1

可得:

$${}_7\Phi_6 \left[ \begin{matrix} aq, \sqrt{aq^2}, -\sqrt{aq^2}, bq, cq, q, q \\ \sqrt{aq}, -\sqrt{aq}, aq^2/b, aq^2/c, aq^2, q^2 \end{matrix}; q, \frac{qa}{bc} \right] \\ = \frac{bc}{qa} \frac{(1-aq/b)(1-aq/c)(1-aq)}{(1-aq^2)(1-b)(1-c)} \left[ \widetilde{\Psi}_q(\frac{aq}{c}) - \widetilde{\Psi}_q(aq) + \widetilde{\Psi}_q(\frac{aq}{b}) - \widetilde{\Psi}_q(\frac{aq}{bc}) \right].$$

又设  $c = \sqrt{a}$ , 则可得:

$${}_5\Phi_4 \left[ \begin{matrix} aq, -\sqrt{aq^2}, bq, q, q \\ -\sqrt{aq}, aq^2/b, aq^2, q^2 \end{matrix}; q, q\sqrt{a}/b \right] \\ = \frac{b}{\sqrt{aq}} \frac{(1-aq/b)(1-aq)(1+\sqrt{a})}{(1-b)(1-a)(1+\sqrt{aq})} \left[ \widetilde{\Psi}_q(\sqrt{aq}) - \widetilde{\Psi}_q(aq) + \widetilde{\Psi}_q(\frac{aq}{b}) - \widetilde{\Psi}_q(\frac{\sqrt{aq}}{b}) \right]$$

$$\begin{aligned}\text{而} \quad \widetilde{\Psi}_q(\sqrt{aq}) - \widetilde{\Psi}_q(aq) &= \sum_{n=0}^{\infty} \frac{q^n(aq - \sqrt{aq})(1-q)}{(1 - \sqrt{aq^{n+1}})(1 - aq^{n+1})}, \\ \widetilde{\Psi}_q\left(\frac{aq}{b}\right) - \widetilde{\Psi}_q\left(\frac{\sqrt{aq}}{b}\right) &= \sum_{n=0}^{\infty} \frac{bq^n(1-q)(\sqrt{aq} - aq)}{(b - \sqrt{aq^{n+1}})(b - aq^{n+1})},\end{aligned}$$

代入后,化简即得:

$$\begin{aligned}_5\Phi_4\left[\begin{matrix} aq, -\sqrt{aq^2}, bq, q, q \\ -\sqrt{aq}, aq^2/b, aq^2, q^2 \end{matrix}; q, \frac{\sqrt{aq}}{b}\right] &= \frac{b(aq-b)(1-aq)(1-q)}{(1-b)(1+\sqrt{aq})} \\ &\quad \left\{ \sum_{n=0}^{\infty} \left[ \frac{1/b}{(1-\sqrt{aq^{n+1}})(1-aq^{n+1})} - \frac{1}{(b-\sqrt{aq^{n+1}})(b-aq^{n+1})} \right] \right\}.\end{aligned}$$

$$\begin{aligned}\text{推论} \quad {}_4\Phi_4\left[\begin{matrix} aq, -\sqrt{aq^2}, q, q \\ -\sqrt{aq}, aq^2, q^2, 0 \end{matrix}; q, \sqrt{aq^2}\right] \\ = \frac{q^n(1-q)(1-aq)}{(1+\sqrt{aq})} \cdot \sum_{n=0}^{\infty} \frac{1}{(1-aq^{n+1})(1-\sqrt{aq^{n+1}})}.\end{aligned}$$

证明 以  $b = \sqrt{a}$  代入定理 3 后,即可得出.

### 参 考 文 献

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## The Summations for Basic Hypergeometric Series

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**Abstract** By using a simple algorithm for the summation of basic hypergeometric series, summation formulas for some basic hypergeometric series are obtained.

**Key words** basic hypergeometric series; summation; gamma function